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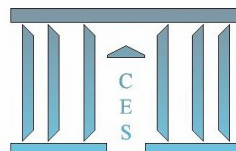
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Anti-monotone risk aversion**

Moez ABOUDA, Elyess FARHOUD

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# Anti-comonotone random variables and Anti-monotone risk aversion \*

Moez Abouda <sup>† ‡</sup>      Elyéss Farhoud <sup>§</sup>

## Abstract

This paper focuses on the study of decision making under risk. We, first, recall some model-free definitions of risk aversion and increase in risk. We propose a new form of behavior under risk that we call anti-monotone risk aversion (hereafter referred to as ARA) related to the concept of anti-comonotony a concept investigated in Abouda, Aouani and Chateauneuf (2008). Note that many research has already been done in this field e.g. through the theory of comonotonicity. We give relationships between comonotone, strict comonotone, anti-comonotone and strict anti-comonotone random variables. Then, after the motivation of ARA, we show that this new aversion is weaker than monotone risk aversion while stronger than weak risk aversion.

**Keywords:** Risk aversion, model-free concepts, comonotone, strict comonotone, anti-comonotone, strict anti-comonotone, Anti monotone risk aversion.

**JEL classification:** D80 - D81 - G12.

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<sup>†</sup>Corresponding author, E-mail: abouda@univ-paris1.fr

<sup>‡</sup>BESTMOD: Institut Supérieur de Gestion de Tunis, 41 Av. de la Liberté -Cité-Bouchoucha. 2000 le Bardo, Tunisie and CES. E-mail: abouda@univ-paris1.fr

<sup>§</sup>Ecole Polytechnique de Tunisie, la Marsa, Tunisie and BESTMOD: Institut supérieur de gestion de Tunis . E-mail: farhoudelyess@yahoo.com

# 1 Introduction

The paper is organized as follows: In Section 2 we give some notations and preliminaries that are useful to our study and we recall rapidly the set of axioms that are usual and natural requirements whatever the attitude towards risk may be. In Section 3, some models of decision under risk are presented with more details. In Section 4, we explicitly give definitions of increase in risk and different form of risk aversion that are model-free e.g, independently of any model of decision under risk and we give some fundamental properties related to the concept of comonotonicity. Here, we establish the relationships that ties comonotone, strict comonotone, anti-comonotone and strict anti-comonotone random variables. In section 5 and 6, which are the focus of our paper, we propose a new form of risk aversion that we call "Anti monotone risk aversion". After the motivation of ARA, we compare it with other forms of risk aversion namely monotone and weak risk aversion. ARA is based on reducing the risk degree of a lottery hence we get what we call a "hedging" and we show that this aversion takes place between monotone risk aversion and weak risk aversion.

## 2 Notations and preliminaries

We suppose that we have a decision-maker faced with choices among risky assets  $X$ , the set  $V$  of such assets consisting of all bounded real random variables defined on a probability space  $(S, \mathcal{A}, P)$ <sup>1</sup> assumed to be sufficiently rich to generate any bounded real-valued random variable.  $S$  is the set of states of nature,  $\mathcal{A}$  is a  $\sigma$ -algebra of events (i.e. of subsets of  $S$ ), and  $P$  is a  $\sigma$ -additive non-atomic probability measure. Let  $V_0$  containing only discrete

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<sup>1</sup>we assume that the decision maker is in a situation of risk. He knows the probability distribution  $P$ , which is exogenous, on the set of states of nature: The set  $(S; \mathcal{A})$  endowed with this probability measure is thus a probability space  $(S; \mathcal{A}; P)$

elements of  $V$ .

**Definition 2.1.** *A family  $\mathcal{A}$  of subsets of the universe  $S$  is called a sigma-algebra if it fulfils the three following properties:*

- $S \in \mathcal{A}$  (i.e,  $S$  itself is an event);
- $E \in \mathcal{A} \Rightarrow \bar{E} \in \mathcal{A}$  (i.e,  $\bar{E}$  is called the complement of the event  $E$ );
- $E_1, E_2, E_3, \dots \in \mathcal{A} \Rightarrow \bigcup_{i \geq 1} E_i \in \mathcal{A}$ .

A risk can be described as an event that may or may not take place, and that brings about some adverse financial consequences. It is thus natural that the modeling of risks uses probability theory. Thus, any  $X$  of  $V$  is then a random variable and has then a probability distribution denoted  $P_X$ . Let  $F_X$ <sup>2</sup> denote the cumulative distribution function of  $P_X$  such that  $F_X(x) = P\{X \leq x\}$ . Even if the distribution function  $F_X$  does not tell us what is the actual value of  $X$ , it thoroughly describes the range of possible values for  $X$  and the probabilities assigned to each of them. Let  $G_X(x) = P(X > x) = 1 - F_X(x)$  be the survival function (also called tail function) and  $E(X)$  the expected value of  $X$ .

For each Decision maker there exists a binary preference relation  $\succeq$  (i.e. a nontrivial weak order) over  $V$ .  $\succeq$  is then transitive and complete. The relation  $\succeq$  is said to be “nontrivial” if there exists  $X$  and  $Y \in V$  such that  $X \succ Y$ ; “complete” if  $\forall X, Y \in V$ ,  $X \succeq Y$  or  $Y \succeq X$  and “transitive” if  $\forall X, Y, Z \in V$ ,  $X \succeq Y$  and  $Y \succeq Z \Rightarrow X \succeq Z$ . Thus for any pair of assets  $X, Y$ ;  $X \succeq Y$  means that  $X$  is weakly preferred to  $Y$  by the DM,  $X \succ Y$  means that  $X$  is strictly preferred to  $Y$  and  $X \sim Y$  means that  $X$  and  $Y$  are

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<sup>2</sup>In words,  $F_X(x)$  represents the probability that the random variable  $X$  assumes a value that is less than or equal to  $x$ .

considered as equivalent by the DM.

First we state three axioms which are usual and natural requirements, whatever the attitude towards risk may be.

**(A.1)  $\succeq$  respects first-order stochastic dominance**

$$\forall X, Y \in V, [P(X \geq t) \geq P(Y \geq t) \forall t \in \mathbb{R}] \Rightarrow X \succeq Y.$$

In words, if, for each amount  $t$  of money, the probability that lottery  $X$  yields more than  $t$  is greater than the probability that lottery  $Y$  yields more than  $t$ , then  $X$  is preferred to  $Y$ . This implies that identically distributed random variables are indifferent to the decision maker.

**(A.2) Continuity with respect to monotone simple convergence**

$$\forall X_n, X, Y \in V$$

$$[X_n \downarrow X, X_n \succeq Y \forall n] \Rightarrow X \succeq Y$$

$$[X_n \uparrow X, X_n \preceq Y \forall n] \Rightarrow X \preceq Y$$

$X_n \downarrow X$  (resp  $X_n \uparrow X$ ) means that  $X_n$  is a monotonic decreasing (resp. monotonic increasing) sequence simply converging to  $X$ .

**(A.3) Monotonicity**

$$[X \geq Y + \varepsilon.S, \varepsilon > 0] \stackrel{3}{\Rightarrow} X \succ Y$$

One can show that any preference relation satisfying the axioms above may be characterized by a unique real number  $c(X)$  to be referred to as *the certainty equivalent* of  $X$  :  $X \sim c(X).S$ , where  $c(\cdot)$  satisfies :

- $X \succeq Y \Leftrightarrow c(X) \geq c(Y)$ .
- $X \geq Y \Rightarrow c(X) \geq c(Y)$  and  $X \geq Y + \varepsilon.S, \varepsilon > 0 \Rightarrow c(X) > c(Y)$ .

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<sup>3</sup>For  $A \in \mathcal{A}$ , De Finetti's use of  $A$  to denote the characteristic function of  $A$  [ $A(s) = 1$  if  $s \in A$ ,  $A(s) = 0$  if  $s \notin A$ ] will be adopted.

- $X_n, X, Y \in V$ ;  $X_n \downarrow X \Rightarrow c(X_n) \downarrow c(X)$  ;  $X_n \uparrow X \Rightarrow c(X_n) \uparrow c(X)$ .
- $X \succeq_{FSD} Y \Rightarrow c(X) \geq c(Y)$ .

Note that the existence of this certainty equivalent is guaranteed by the continuity and monotonicity assumptions, and it can be used as a representation of  $\succeq$ .

### 3 Models of decision under risk

Decision theory under risk has always for objective to describe agent's behaviors using several uncertain perspectives and assuming that every agent is characterized by his own preferences. Since, to completely describe these preferences is difficult, we aim at representing them. Thus, by assigning a numerical value to every uncertain perspective, we can easily rank one agent's preferences. In fact, the resort to a representative function of preferences form, appears to be, since a long time ago, the usual method to describe behaviors in a uncertainty context. The evident interest is to allow the direct integration of these data into a formalized model and, by extension, to understand the optimization process which is subjacent to every decision. Nevertheless, the determination of the preference representative function have to rely on an axiomatic foundation. Using these axioms, we will derive a precise specification of the value function. Let us first introduce the classical model of decision under risk, the expected utility model.

#### 3.1 The expected utility model (EU)

The Expected Utility (EU) model, first introduced in the seminal work of von Neumann and Morgenstern (1944)[43] is the classical model of decision under risk, which furthermore satisfies the central sure thing principle of Savage,

**(A.4) Sure thing principle:**

Let  $X, Y \in V_0$  such that  $\mathcal{L}(X) = (x_1, p_1; \dots; x_i, p_i; \dots; x_n, p_n)$  and  $\mathcal{L}(Y) = (y_1, p_1; \dots; y_i, p_i; \dots; y_n, p_n)$ , with  $x_1 \leq \dots \leq x_i \leq \dots \leq x_n$ ,  $p_i \geq 0$ ,  $y_1 \leq \dots \leq y_i \leq \dots \leq y_n$ ,  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$  and suppose that for a certain  $i_0$ , we have  $x_{i_0} = y_{i_0}$ .

The axiom tells us that the preference between  $X$  and  $Y$  are unchanged if we replace  $x_{i_0}$  and  $y_{i_0}$  by a common  $t \in \mathbb{R}$ .

In this model, preferences can be represented (see Fishburn and Wakker (1995)[28]), due both to the independence axiom and the von Neumann Morgenstern (vNM) theorem, by the expected utility denoted  $E(u(X))$  such that:

$$E(u(X)) = \int_{-\infty}^0 [(P(u(X) > t)) - 1] dt + \int_0^{\infty} (P(u(X) > t)) dt \quad (1)$$

where  $u$  is the utility function of von Neumann Morgenstern;  $u : \mathbb{R} \rightarrow \mathbb{R}$ , is continuous, strictly increasing and unique up to an affine increasing transformation. The best decision being the one maximizing this Expected Utility. For a discrete random variable  $X \in V_0$  ( $X$  is a lottery) with law of probability  $\mathcal{L}(X) = (x_1, p_1; \dots; x_i, p_i; \dots; x_n, p_n)$ , with  $x_1 < \dots < x_i < \dots < x_n$ ,  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$ , the formula (1) reduces to:

$$E(u(X)) = \sum_{i=1}^n p_i \cdot u(x_i) \quad (2)$$

Even if the EU model has the advantage to be parsimonious (nevertheless any kind of risk aversion is characterized by a concave utility function), many observed economic behaviors cannot be explained in the framework of this model. Consequently, we will present, next, the Rank-Dependent Expected Utility model(RDU), a more general model, less parsimonious but more explanatory. But above all, we rapidly investigate the Dual Yaari's model which is proved to be more flexible than EU theory.



**Remark 3.1.** *Note that in the framework of expected utility, it's impossible to distinguish the agent's attitude towards risk from the agent's attitude towards wealth (since  $u$  is concave). By contrast, in the dual theory of choice under risk proposed by Yaari (1987), we see easily that these two notions are kept separate from each other.*

## 3.2 The Yaari model

One of the most successful nonexpected utility models is the dual theory of choice under risk due to Yaari (1987). In this model, **the comonotone independence axiom** will be substituted to **the sure thing principle axiom**. Let us first define comonotonicity.

### Definition(A.4)' Comonotonicity

Two real-valued functions  $X$  and  $Y$  on  $S$  are *comonotone* if for any  $s$  and  $s' \in S$ ,

$$[X(s) - X(s')] [Y(s) - Y(s')] \geq 0.$$

### Axiom (A.4)' Comonotone Independence

$[X \text{ and } Z \text{ are comonotone, } Y \text{ and } Z \text{ are comonotone, } X \sim Y] \Rightarrow X + Z \sim Y + Z$

Under axioms (A.1),(A.2),(A.3) and (A.4)', Chateauneuf (1994) showed that the function  $c(X)$  which represent preferences is not other than the certainty equivalent of Yaari(1987):

$$c(X) = \int_{-\infty}^0 [f(P(X) > t)) - 1] dt + \int_0^{\infty} f(P(X) > t))dt \quad (3)$$

Where  $f : [0, 1] \rightarrow [0, 1]$  is the probability transformation function, is continuous, increasing and such that  $f(0) = 0$  and  $f(1) = 1$ .

We can interpret the function  $f$  differently as the probability distortion or

perception function since it is an adjustment of the underlying objective probability due to the subjective risk perception of the decision maker.

For a discrete random variable  $X \in V_0$  with law of probability  $\mathcal{L}(X) = (x_1, p_1; \dots; x_i, p_i; \dots; x_n, p_n)$ , with  $x_1 < \dots < x_i < \dots < x_n$ ,  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$ , the formula (3) reduces to:

$$c(X) = x_1 + \sum_{i=2}^n \left[ (x_i - x_{i-1}) \cdot f\left(\sum_{j=i}^n p_j\right) \right] \quad (4)$$

This theory, while eliminating some of expected utility's drawbacks, shares with expected utility the completeness assumption: the decision maker must be able to rank any pair  $(X, Y)$  of lotteries.

### 3.3 The Rank Dependent Expected Utility model(RDU)

In order to take into account the paradoxes of Allais (1953)[7] and to separate perception of risk from the valuation of outcomes (which ones are taken into account by the same tool, the utility function in EU theory) an alternative theory the rank dependent expected utility(RDU) first elaborated by Quiggin (1982)[35] under the denomination of "Anticipated Utility" has been developed since the early eighties. The Rank Dependent Expected Utility model is the most widely used, and arguably the most empirically successful, generalization of the expected utility model(EU). Variants of this model are due to Yaari(1987)[46] and Allais (1988)[8]. More general axiomatizations can be found in Wakker (1994)[44], Chateauneuf (1999)[13].

Let us recall that a RDU DM weakly prefers  $X$  to  $Y$ ,  $X, Y \in V$  if and only if  $E(u(X)) \geq E(u(Y))$ , where  $E(u(Z))$  is defined for every  $Z \in V$  by :

$$E(u(Z)) = \int_{-\infty}^0 [f(P(u(Z) > t)) - 1] dt + \int_0^{\infty} f(P(u(Z) > t)) dt \quad (5)$$

Roughly speaking, in RDU theory, individuals' preferences over risky prospects are represented by the mathematical expectation of a utility function  $u$  with respect to a transformation  $f$  of the outcomes cumulative probabilities.  $u$  utility of wealth,  $u : \mathbb{R} \rightarrow \mathbb{R}$  is assumed to be cardinal (i.e., defined up to a positive affine transformation), strictly increasing and continuous.

$f : [0, 1] \rightarrow [0, 1]$  the probability transformation function (as in Yaari) is assumed to be strictly increasing, continuous and such that  $f(0) = 0$  and  $f(1) = 1$ . Note that, in this model, the transformation function  $f$  is defined under cumulative probabilities rather than simple probabilities. That's why the ranking of outcomes is fundamental (which explains the denomination of Rank Dependent Expected Utility).

For a discrete random variable  $Z$  with probability law

$\mathcal{L}(Z) = (z_1, p_1; \dots; z_k, p_k; \dots; z_n, p_n)$ , where  $z_1 \leq z_2 \leq \dots \leq z_n$ ,  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$ , the formula (5) reduces to :

$$E(u(Z)) = u(z_1) + \sum_{i=2}^n (u(z_i) - u(z_{i-1})) \left[ f\left(\sum_{j=i}^n p_j\right) \right] \quad (6)$$

Such a formula is meaningful : the DM takes for sure the utility of the minimum payoff, and then add the successive possible additional increments of utility weighted by his personal perception of the related probability.

RDU theory reduces to EU theory if  $f$  is the identity function, and RDU theory reduces to the dual theory of Yaari if  $u$  is the identity function. Unlike EU, RDU preferences allows us to discriminate amongst different notions of risk aversion.

## 4 Model-free concepts of risk aversion

Along this section, we rapidly give different concepts and definitions of some risk aversion and increase in risk that are independent to any model (e.g. model-free). Let us first define stochastic orders.

**Definition 4.1. *First order stochastic dominance***<sup>4</sup>

Let  $X$  and  $Y$  the elements of  $V$ ,  $X$  is said to dominate  $Y$  for the first order stochastic dominance to be denoted  $(X \succeq_{FSD} Y)$  if :

$$Pr[X > t] \geq Pr[Y > t] \quad \forall t \in \mathbb{R}$$

$$i.e., \quad F_X(t) \leq F_Y(t) \quad \forall t \in \mathbb{R}$$

The second concept is weaker than FSD and is called second-order stochastic dominance (SSD).

**Definition 4.2. *Second order stochastic dominance [Rothschild and Stiglitz, 1970[40]]***

Let  $X, Y \in V$ ,  $X$  is said to dominate  $Y$  for the second order stochastic dominance to be denoted  $(X \succeq_{SSD} Y)$  if :

$$\int_{-\infty}^x F_X(t)dt \leq \int_{-\infty}^x F_Y(t)dt \quad \forall x \in \mathbb{R}.$$

**Definition 4.3. *Weak Risk Aversion [Arrow(1965)[10], Pratt(1964)[34]]***

A DM exhibits Weak Risk Aversion (WRA) if, for any random variable  $X$  of  $V$ , he prefers to the random variable  $X$ , its expected value  $E(X)$  with certainty:

$$\forall X \in V, \quad E(X) \succeq X$$

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<sup>4</sup>Stochastic dominance is a term which refers to a set of relations that may hold between a pair of distribution

To introduce the notion of strong risk aversion, we have first to define Mean Preserving Spread based on second order stochastic dominance already defined.

**Definition 4.4.** *Mean Preserving Spread [Rothschild and Stiglitz (1970)[40]]*

For  $X$  and  $Y$  with the same mean,  $Y$  is a general mean preserving increase in risk or Mean Preserving Spread (MPS) of  $X$  if :

$$\int_{-\infty}^x F_X(t)dt \leq \int_{-\infty}^x F_Y(t)dt, \forall x \in \mathbb{R}$$

$$E(X) = E(Y) \text{ and } X \text{ SSD } Y \Rightarrow Y \text{ MPS } X$$

**Remark 4.5.** *This notion of increase in risk is considered as a special case, for equal means, of second order stochastic dominance and it could be explained by the fact that the more risky  $Y$  is obtained by adding a noise  $Z^5$  to  $X$ .*

**Definition 4.6.** *Strong Risk Aversion [Hadar and Russell (1969)[30], Rothschild and Stiglitz (1970)[40]]*

A DM exhibits Strong Risk Aversion (SRA) if for any pair of random variables  $X, Y \in V$  with  $Y$  being a Mean Preserving Spread of  $X$ , he always prefers  $X$  to  $Y$  :

$$\forall X, Y \in V, Y \text{ MPS } X \Rightarrow X \succeq Y$$

Quiggin (1992) introduced an alternative notion of monotone (mean-preserving) increase in risk, defined in terms of co-monotonic random variables instead of mean-preserving spreads. However, the notion of strong

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<sup>5</sup> $E(Z | X) = 0$

risk aversion is always considered as too strong by some DMs. As a consequence, Quiggin(1991) proposed the weaker notion called monotone risk aversion based on comonotonicity.

**Definition 4.7. Mean Preserving Monotone Spread [Rothschild and Stiglitz (1970)[40], Quiggin (1991)[36]]**

For  $X, Y \in V$ , the distribution of  $Y$  is a monotone increase in risk of the distribution of  $X$ , or,  $Y$  is a mean preserving monotone spread of  $X$ , if  $\exists Z \in V$  such that  $E(Z) = 0$ ,  $Z$  and  $X$  are comonotone and  $Y =_d {}^6X + Z$ . Thus,  $X$  is said to be less risky than  $Y$  for the monotone risk order denoted  $X \succeq_M Y$ .

**Definition 4.8. Monotone Risk Aversion [Quiggin (1991)[36]]**

A DM is monotone risk averse if for any  $X, Y \in V$  with equal means such that  $Y$  is a monotone mean preserving spread of  $X$ , the DM weakly prefers  $X$  to  $Y$ .

$$i.e., \forall X, Y \in V, \quad X \succeq_M Y \Rightarrow X \succeq Y$$

Abouda and Chateauneuf (2002)[3], Abouda (2008)[1] have introduced a new concepts of risk aversion namely symmetrical monotone risk aversion.

**Definition 4.9. Symmetrical Monotone Risk Order [Abouda and Chateauneuf 2002[4]]**

Let  $X, Y \in V$ ,  $X$  is less risky than  $Y$  for the symmetrical monotone risk order denoted  $X \succeq_{SM} Y$ , if there exists  $Z \in V$  such that  $E(Z) = 0$ ,  $Z$  comonotone with  $X$ ,  $Z =_d -Z$  and  $Y =_d X + Z$ .

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<sup>6</sup> $Y$  has the same probability distribution than  $X + Z$

**Definition 4.10. Symmetrical Monotone Risk Aversion [Abouda and Chateauneuf 2002[4]]**

A DM is said to be symmetrical monotone risk averse denoted SMRA if:

$$\forall X, Y \in V, \quad X \succeq_{SM} Y \Rightarrow X \succeq Y$$

**Definition 4.11. Preference For Perfect Hedging [Abouda and Chateauneuf 2002[3], Abouda 2008[1]]**

The definition of preference for perfect hedging can take one of the three following assertions:

- (i)  $[X, Y \in V, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq X \text{ or } Y.$
- (ii)  $[X, Y \in V, X \succeq Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y.$
- (iii)  $[X, Y \in V, X \sim Y, \alpha \in [0, 1], \alpha X + (1 - \alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y.$

**Remark 4.12.** Preference for perfect hedging means that if the decision maker can attain certainty by a convex combination of two assets, then he prefers certainty to one of these assets.

Chateauneuf and Tallon (2002)[21], Chateauneuf and Lakhnati (2007)[20] have introduced a generalization of preference for perfect hedging which is called preference for sure diversification.

**Definition 4.13. Preference For Sure Diversification (Chateauneuf and Tallon (2002)[21])**

$\succeq$  exhibits preference for sure diversification if for any  $X_1, \dots, X_n \in V$  ;  $\alpha_1, \dots, \alpha_n \geq 0$  such that  $\sum_{i=1}^n \alpha_i = 1$  and  $a \in \mathbb{R}$

$$[X_1 \sim X_2 \sim \dots \sim X_n \text{ and } \sum_{i=1}^n \alpha_i X_i = a] \Rightarrow a \succeq X_i, \quad \forall i$$

.

**Remark 4.14.** *Preference for sure diversification means that if the decision maker can attain certainty by a convex combination of equally desirable assets, then he prefers certainty to any of these assets.*

## 5 From comonotonicity to Anti-comonotonicity

Two random variables are comonotone if they move in the same directions. The notion of comonotonicity can be of great use for finance, insurance and actuarial issues since it can lead to an increase of the risk of a random variable. Hence, the necessity of anti-comonotonicity which plays a crucial role in terms of hedging. In our context, we show how it contributes to reduce the risk of a random variable and may even lead to a perfect hedging. In what follows, different notions of comonotonicity that are dispersed in a wide range of literature are reviewed. We gather different definitions and properties that have been given in various contexts then we introduce new ones namely strict comonotonicity and strict anti-comonotonicity.

**Definition 5.1. *Comonotonicity*** [Yaari (1987), Schmeidler (1989)]

*Two real-valued functions  $X$  and  $Y$  on  $S$  are comonotone if for any  $s$  and  $s' \in S$ ,*

$$[X(s) - X(s')] [Y(s) - Y(s')] \geq 0.$$

**Remark 5.2.** *Note that comonotonicity is not a transitive relation because constant functions are comonotone with any function. Consistent with the usual conventions, random variables are said to be comonotone if they are comonotone functions almost everywhere.*

**Example 5.3.** *Let  $X, Y$  and  $Z \in V$  with  $Y$  constant and assume that there exist two states of nature  $s$  and  $s'$  in  $S$  such that:*



$X(s) > X(s')$  and  $Z(s) < Z(s')$ .

We see easily that  $X$  is comonotone with  $Y$  and  $Y$  is comonotone with  $Z$  but  $X$  is not comonotone with  $Z$ .

Now, we introduce a new definition related to the concept of comonotonicity which is transitive.

**Definition 5.4. Strict comonotonicity**

Two real-valued functions  $X$  and  $Y$  on  $S$  are strictly comonotone if for any  $s$  and  $s' \in S$ ,

$$X(s) > X(s') \Leftrightarrow Y(s) > Y(s').$$

**Remark 5.5.** If  $X$  and  $Y$  are strictly comonotone then they are comonotone.

Now, we give a first definition of anti-comonotonicity as introduced in Abouda, Aouani and Chateauneuf (2008)[2] .

**Definition 5.6. Anti comonotonicity**

Two real-valued functions  $X$  and  $Y$  on  $S$  are anti-comonotone if for any  $s$  and  $s' \in S$ ,

$$[X(s) - X(s')] [Y(s) - Y(s')] \leq 0.$$

**Remark 5.7.** Note that if  $X$  is anti-comonotone with  $Y$  and  $Y$  is anti-comonotone with  $Z$  then,  $X$  and  $Z$  need not be comonotone. Such a result is obtained, for example, because a constant random variable is anti-comonotone with all random variables.

As it was done earlier with comonotonicity and strict comonotonicity, we bring a new concept related to anti-comonotonicity that we call strict anti-comonotonicity.

**Definition 5.8.** *Strict anti-comonotonicity*

Two real-valued functions  $X$  and  $Y$  on  $S$  are strictly anti-comonotone if for any  $s$  and  $s' \in S$ ,

$$X(s) > X(s') \implies Y(s) < Y(s')$$

$$\text{and } X(s) = X(s') \implies Y(s) = Y(s').$$

$$\text{i.e., } X(s) > X(s') \iff Y(s) < Y(s').$$

**Remark 5.9.** If  $X$  and  $Y$  are strictly anti-comonotone then they are anti-comonotone.

We have, thus, the following property:

**Property 5.10.** Let  $X, Y$  and  $Z \in V$ .

- i) If  $X$  is strictly anti-comonotone with  $Y$  and  $Y$  is strictly anti-comonotone with  $Z$  then  $X$  is strict comonotone with  $Z$ .
- ii) If  $X$  is strictly anti-comonotone with  $Y$  and  $Y$  is strictly comonotone with  $Z$  then  $X$  is strictly anti-comonotone with  $Z$ .
- iii) If  $X$  is comonotone with  $Y$  and  $Y$  is strictly comonotone with  $Z$  then  $X$  is comonotone with  $Z$ .

**Proof.**

- i) Suppose that  $X$  is strictly anti-comonotone with  $Y$  and  $Y$  is strictly anti-comonotone with  $Z$ .

Let  $s, s' \in S$

$$X(s) > X(s') \quad \underbrace{\Longleftrightarrow}_{\text{strict anti-com}} \quad Y(s) < Y(s') \quad \underbrace{\Longleftrightarrow}_{\text{strict anti-com}} \quad Z(s) > Z(s').$$

Then  $X$  is strict comonotone with  $Z$ .

- ii) Suppose that  $X$  is strictly anti-comonotone with  $Y$  and  $Y$  is strictly comonotone with  $Z$

Let  $s, s' \in S$

$$X(s) > X(s') \quad \underbrace{\Longleftrightarrow}_{\text{strict anti-com}} \quad Y(s) < Y(s') \quad \underbrace{\Longleftrightarrow}_{\text{strict com}} \quad Z(s) < Z(s').$$

Then  $X$  is strictly anti-comonotone with  $Z$ .

- iii) Suppose that  $X$  is comonotone with  $Y$  and  $Y$  is strictly comonotone with  $Z$ .

Let  $s, s' \in S$

$$X(s) < X(s') \quad \underbrace{\implies}_{\text{com}} \quad Y(s) \leq Y(s') \quad \underbrace{\implies}_{\text{strict com}} \quad Z(s) \leq Z(s').$$

Then  $X$  is comonotone with  $Z$ .

□

### Example 5.11.

Assume that we have the following random variables defined over the set  $S$ , set of states of nature, by:

	$s_1$	$s_2$	$s_3$	$s_4$
$X$	1	2	3	4
$X_1$	-2	0	0	2
$X_2$	-2	-1	1	3
$X_3$	4	2	-1	-3
$X_4$	5	2	2	1
$C$	2	2	2	2

Clearly,  $X$  and  $X_1$  are comonotone but not strictly comonotone.

$X$  and  $X_2$  are strictly comonotone while  $X$  and  $X_3$  are strictly anti-comonotone.

$X$  and  $X_4$  are anti-comonotone and not strictly anti-comonotone.

Nevertheless, constant random variables ( $C$ ) does not show any relation of strict comonotonicity or strict anti-comonotonicity with other random variables.

## 6 Anti monotone risk aversion in model free

In this paper, using anti comonotonicity, we define a new form of behavior which performs a useful role in modeling behavior under risk. This aversion to risk is called "Anti-monotone risk aversion" (ARA). Thus, a DM is said to be anti monotone risk averse if  $X$  is weakly preferred to  $Y$  when  $X$  is less risky than  $Y$  for the anti monotone risk order denoted  $X \succeq_{AMO} Y$ . In what follows, we first, define anti monotone risk order  $AMO$  then we discuss its properties and motivations. Finally, we show that ARA takes place between weak and monotone risk aversion.

### Definition 6.1. Anti-monotone risk order

Let  $X, Y \in V$ ,  $X$  is more risky than  $Y$  for the anti-monotone risk order denoted  $Y \succeq_{AMO} X$ , if  $X$  and  $Y$  are comonotone and there exists  $Z \in V$  strictly anti-comonotone with  $X$  such that  $E(Z) = 0$  and  $Y =_d X + Z$ .

### Definition 6.2. Anti-monotone risk aversion

A DM is anti-monotone risk averse denoted ARA if:

$$\forall X, Y \in V, \quad X \succeq_{AMO} Y \implies X \succeq Y.$$

## 6.1 Motivations of Anti-monotone risk aversion

Let us first give the following example

**Example 6.3.**

	$s_1$	$s_2$	$s_3$	$s_4$
$Pr$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$X$	2	2	2	2
$Z$	-2	-1	1	2
$Y = X + Z$	0	1	2	4

$Z$  is a -zero mean- random variable defined over  $S$  and is anti-comonotone with  $X$ .

We see here that risk has increased .

This explains why we use strict anti comonotonicity rather than anti comonotonicity in the definition of ARA.

Let us give now a second example

**Example 6.4.**

	$s_1$	$s_2$	$s_3$	$s_4$
$Pr$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$X$	1	2	3	4
$Z_1$	1	0.5	-0.5	-1
$Z_2$	10	5	-5	-10
$Z_3$	1.5	0.5	-0.5	-1.5

$Z_i$  are -zero mean- random variables defined over  $S$ :  $X$  is now strict anti-comonotone with all  $Z_i$ .

Let  $Y_i = X + Z_i$ .

	$s_1$	$s_2$	$s_3$	$s_4$
$Pr$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$X$	1	2	3	4
$Y_1 = X + Z_1$	2	2.5	2.5	3
$Y_2 = X + Z_2$	11	7	-2	-6
$Y_3 = X + Z_3$	2.5	2.5	2.5	2.5

Adding  $Z_1$  to  $X$ , leads to a decrease of the risk and we got what we call a "hedging".

But if we add  $Z_2$  to the same random variable  $X$ , this time, the risk increases and we obtain large gaps between the outcomes of  $Y_2$  in the opposite direction.

In definition **6.1**, we have assumed  $X$  to be comonotone with  $Y$  while this does not match with  $X$  and  $Y_2$  since, in our case,  $X$  and  $Y_2$  are not comonotone.

Nevertheless,  $Y_3 = X + Z_3$  is called a situation of perfect hedging, such a result can be reached by adding  $Z = E(X) - X$  to any random variable  $X$ <sup>7</sup>.

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<sup>7</sup>We can see easily that  $E(Z) = 0$  and  $Z$  is strict anti-comonotone with  $X$ .  $Y = E(X)$  and then comonotone with  $X$ .

## 6.2 Comparison between weak, monotone and anti monotone risk aversion

As we will show in this part, these three kinds of behavior under risk are logically related so that monotone risk aversion implies anti-monotone risk aversion which in turn implies weak risk aversion. Abouda and Chateauneuf [4] have characterized *SMRA* in different models of choice under risk and they proved that *SMRA* is weaker than *MRA*. In a similar way, we establish the relation that ties monotone, anti monotone and weak risk aversion. Theorem 6.5 and 6.6 illustrate our contribution.

### Theorem 6.5.

*Monotone risk aversion  $\implies$  Anti-monotone risk aversion.*

#### Proof.

Let  $X, Z \in V$  such that  $Z$  is strictly anti-comonotone with  $X$ ,  $E(Z) = 0$  and  $X + Z$  comonotone with  $X$ .

Given that  $-Z$  is strictly anti-comonotone with  $Z$  then, by i) of property 5.10, we have  $-Z$  is strictly comonotone with  $X$  and by iii) we get  $-Z$  is comonotone with  $X + Z$ .

Definition 4.7 gives  $X + Z \succeq_M X + Z + (-Z)$ .

Then, according to definition 4.8, we have  $X + Z \succeq X$ . □

### Theorem 6.6.

*Anti-monotone risk aversion  $\implies$  Weak risk aversion.*

#### Proof.

Let  $Z = E(X) - X$

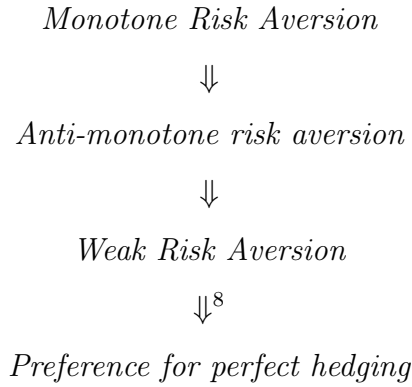
We can see easily that  $Z$  is strictly anti-comonotone with  $X$ ,  $E(Z) = 0$  and  $X + Z$  comonotone with  $X$ .

Then, by definition **6.1**, one can obtain  $X + Z \succeq_{AMO} X$ ,

Then, following our hypothesis and definition **6.2**, we have  $E(X) \succeq X$ .  $\square$

To summarize, the relation that ties anti-monotone risk aversion, weak risk aversion and monotone risk aversion is given by the following theorem.

**Theorem 6.7.**



while the reciprocal assertions are not necessarily true.

## Conclusion

In this paper, definitions of increases in risk, and comparative degrees of aversion to such increases in risk have been presented independently to any model of decision under risk. In this context, our contribution is to propose, using anti-comonotonicity, a new form of risk aversion namely "Anti monotone risk

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<sup>8</sup>For more details on other relationships of risk aversions see Abouda and Farhoud [5].



aversion". We have shown that ARA is intimately related to Hedging effects. Note that the latter is stronger than weak risk aversion while weaker than monotone risk aversion. We give relationships between comonotone, strict comonotone, anti-comonotone and strict anti-comonotone random variables and we discuss some properties related to comonotonicity. Our results show that this new aversion to risk contributes in the adjustment of risk since it reduces the risk associated to a random variable. It participates to the realization of the hedging process through the hypothesis it requires namely strict anti-comonotonicity and comonotonicity.

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